

Discontinuities and collision of gravitational waves in string theory ^{*}

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Abstract

We examine here discontinuities in the metric, the antisymmetric tensor and the dilaton field which are allowed by conformal invariance. We find that the surfaces of discontinuity must necessarily be null surfaces and shock and impulsive waves are both allowed. We employ our results for the case of colliding plane gravitational waves and we discuss the $SL(2, \mathbb{R}) \times SU(2)/\mathbb{R} \times \mathbb{R}$ WZW model in the present perspective. In particular, the singularities encountered in this model may be viewed as the result of the mutual focusing of the colliding waves.

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An interesting class of solutions in general relativity possessing remarkable properties consists of plane gravitational waves [1]. These solutions constitute their own linearized approximation and thus may be viewed as the exact classical analogs of gravitons. The space-time of such waves is complete and it does not possess global Cauchy surfaces [1],[2]. These solutions can be extended to exact string solutions [3]. Exact plane wave string backgrounds have been obtained by employing WZW models based on non-semisimple groups as well as various gaugings of the latter [4]. These models can also be obtained from semisimple WZW models by taking special singular limits [2],[5].

On the other hand, waves can, in principle, collide and one may ask if it is possible a particular string background to be interpreted as the result of the collision of plane waves, at least in a semiclassical approximation. Before trying to answer this question, let us recall some known results.

The existence of wave-like solutions in Einstein gravity it has long been recognized. There exist mainly two types of waves, shock and impulsive ones [2]. Shock waves have discontinuities in the Riemann tensor, (C^1 -metric), while impulsive ones have a δ -function profile in the curvature (C^0 -metric). Shock wave backgrounds have also been discussed in string theory [6] as well as in field theory because they play an important role in scattering process in ultra-high energies [7],[8]. On the other hand, impulsive waves may be considered as the most “elementary” ones and they have been studied in connection with the motion of massive particles along the horizon of a black-hole background [9],[10].

However, due to the non-linearity of the field equations, wave solutions cannot superposed except in the weak field limit. In fact, nowhere the non-linear character of Einstein gravity shows up more clearly than in the collision of gravitational waves [11]. Unlike a linear theory, i.e., classical electrodynamics, where waves pass straight through each other, in general relativity waves necessarily tend to focus. For plane waves the focus usually appears as a singularity in space-time [11]–[13]. This seems to be a generic feature of the collision provided that the waves are sufficiently strong to produce serious focusing.

We will discuss below the necessary conditions to have discontinuities in the target space in string theory in such a way that their presence do not affect the beta-function equations. We will see that the discontinuities must be across null hypersurfaces and thus backgrounds resulting from linear superposition of independent shock or impulsive waves with parallel propagating wave-fronts are exact as well. We will also discuss the case of the collision of opposite moving plane waves. For this case, we consider the $SL(2, \mathbb{R}) \times SU(2)/\mathbb{R} \times \mathbb{R}$ WZW model [15] and we

show that it can be interpreted, to leading order, as the resulting space-time of the collision of two such waves.

String propagation in a non-trivial background is described by the 2D σ -model action

$$S = \frac{1}{4\pi\alpha'} \int dzd\bar{z} \left((G_{\mu\nu}(X) + B_{\mu\nu}(X)) \partial X^\mu \bar{\partial} X^\nu + \alpha' R^{(2)} \Phi(X) \right), \quad (1)$$

where $G_{\mu\nu}(X)$, $B_{\mu\nu}(X)$ and $\Phi(X)$ are the target metric, the antisymmetric tensor and the dilaton field, respectively. At one-loop level of the coupling constant α' , conformal invariance requires

$$R_{\mu\nu} - \frac{1}{4} H_{\mu\kappa\lambda} H_\nu{}^{\kappa\lambda} + 2\nabla_\mu \nabla_\nu \Phi = 0, \quad (2)$$

$$\nabla_\mu (e^{-2\Phi} H^{\mu\nu\rho}) = 0, \quad (3)$$

$$\frac{2\delta c}{3} - R + \frac{1}{12} H^2 - 4\nabla^2 \Phi + 4(\nabla\Phi)^2 = 0, \quad (4)$$

where $H_{\mu\nu\rho} = \nabla_{[\mu} B_{\nu\rho]}$ is the field strength of the antisymmetric field and δc is the central charge deficit. One assumes that the vacuum is of the form $M^4 \times K$ where M^4 is the Minkowski space-time represented by a free theory with $c = 4$ and K is the internal space corresponding to a conformal field theory of appropriate central charge. By replacing M^4 by another four-dimensional target space N representing again a $c = 4$ conformal field theory, one may obtain other backgrounds for consistent string propagation. Such backgrounds satisfy Eqs. (2–4) with $\delta c = 0$ and may be realized, for example, as gravitational waves or cosmological solutions. All possible spaces N considered so far have been assumed to be endowed with metric, antisymmetric tensor and dilaton fields which are continuous everywhere and have continuous derivatives as well. Here, we will discuss the possibility of solving Eqs. (2–4) for four-dimensional space-times N in which we will allow discontinuities of the metric, antisymmetric tensor and dilaton field across appropriate hypersurfaces.

To begin with, let us consider the antisymmetric three-form field $H_{\mu\nu\rho}$ in N and let us suppose that it has a finite discontinuity across a hypersurface Σ defined by the equation $u(X^\mu) = 0$. We may represent such a field in terms of distributions over a suitable set of test functions as

$$H_{\mu\nu\rho} = H_{\mu\nu\rho}^{(0)} + h_{\mu\nu\rho} \theta(u), \quad (5)$$

where $H_{\mu\nu\rho}^{(0)}$ and $h_{\mu\nu\rho}$ are C^0 and piecewise C^1 and $\theta(u)$ is the Heaviside step function distribution. Thus, we have

$$\begin{aligned} H_{\mu\nu\rho} &= H_{\mu\nu\rho}^{(0)} \quad , \quad u < 0, \\ H_{\mu\nu\rho} &= H_{\mu\nu\rho}^{(0)} + h_{\mu\nu\rho} \quad , \quad u > 0. \end{aligned}$$

By using the notation

$$[f] = f^+ - f^-,$$

where $f^+(f^-)$ is the limit of f as one approaches the surface Σ from the left (right) $u < 0(u > 0)$, we may express Eq. (5) as

$$[H_{\mu\nu\rho}] = h_{\mu\nu\rho}.$$

The closeness of $H_{\mu\nu\rho}$ and Eq. (3) gives

$$\begin{aligned} \partial_{[\kappa} H_{\mu\nu\rho]}^{(0)} + \partial_{[\kappa} h_{\mu\nu\rho]} \theta(u) + h_{[\mu\nu\rho} u_{\kappa]} \delta(u) &= 0, \\ \nabla^\mu (e^{-2\Phi} H_{\mu\nu\rho}) + \nabla^\mu (e^{-2\Phi} h_{\mu\nu\rho}) \theta(u) + e^{-2\Phi} h_{\mu\nu\rho} u^\mu \delta(u) &= 0, \end{aligned}$$

where $u_\mu = \partial u / \partial x^\mu$ and $\delta(u)$ is the Dirac δ -function. In order these equations to hold, the conditions

$$h_{[\mu\nu\rho} u_{\kappa]} = 0, \tag{6}$$

$$h_{\mu\nu\rho} u^\mu = 0. \tag{7}$$

have to be satisfied. We may express $h_{\mu\nu\rho}$ as the dual of a vector h^μ

$$h_{\mu\nu\rho} = \epsilon_{\mu\nu\rho\lambda} h^\lambda,$$

($\epsilon_{0123} = +1$) and, consequently, Eqs. (6,7) are written as

$$\begin{aligned} h^\mu u_\mu &= 0, \\ h^\lambda \epsilon_{[\lambda\mu\nu\rho} u_{\kappa]} &= 0. \end{aligned}$$

The solution to the above equations is

$$\begin{aligned} h^\mu &= h u^\mu, \\ u^\mu u_\mu &= 0, \end{aligned} \tag{8}$$

where h is a scalar. As a result, the surfaces of discontinuity must be null surfaces and the possible discontinuities of $H_{\mu\nu\rho}$ are of the form

$$h_{\mu\nu\rho} = h \epsilon_{\mu\nu\rho\lambda} u^\lambda. \tag{9}$$

A discontinuity of the type (5) in the H -field could only be emerged from a corresponding one in the antisymmetric tensor of the form

$$B_{\mu\nu} = B_{\mu\nu}^{(0)} + b_{\mu\nu} \theta(u). \tag{10}$$

The field strength is then given by

$$H_{\mu\nu\rho} = H_{\mu\nu\rho}^{(0)} + \partial_{[\mu} b_{\nu\rho]} \theta(u) + b_{[\mu\nu} u_{\rho]} \delta(u), \quad (11)$$

and by comparing Eqs. (5,11), we find that

$$h_{\mu\nu\rho} = \partial_{[\mu} b_{\nu\rho]}, \quad (12)$$

$$b_{[\mu\nu} u_{\rho]} = 0. \quad (13)$$

Consequently, the discontinuities in the antisymmetric tensor is of the form

$$b_{\mu\nu} = b_{[\mu} u_{\nu]}. \quad (14)$$

Let us now examine the type of discontinuities of the dilaton field $\Phi(X)$. We will assume that $\Phi(X)$ is continuous while its derivatives may have finite jumps across the hypersurface Σ of the form

$$\partial_\mu \Phi = \partial_\mu \Phi^{(0)} + \phi_\mu \theta(u). \quad (15)$$

To find the explicit form of the discontinuity ϕ_μ , we expand the dilaton $\Phi(x)$ in the neighborhood of Σ as

$$\begin{aligned} \Phi(x^+) &= \Phi_0 + \Phi'_+ u + \frac{1}{2} \Phi''_+ u^2 + \dots, \\ \Phi(x^-) &= \Phi_0 + \Phi'_- u + \frac{1}{2} \Phi''_- u^2 + \dots, \end{aligned} \quad (16)$$

where primes (') denote derivatives with respect to u on Σ . By differentiating the above expressions, we find that ϕ_μ is proportional to u_μ , i.e.,

$$\phi_\mu = \phi u_\mu.$$

It is easy then to check that this type of discontinuity is compatible with Eq. (4) since δ -terms coming from differentiation of (15) disappear because of the nullity of u_μ .

Let us now turn to Eq. (2). One may expect that the appropriate conditions across surfaces are the Lichnerowicz ones. The latter postulate continuation of the metric and its first derivatives, i.e., the metric is considered to be C^1 and piecewise C^2 . Consequently, the Riemann tensor is piecewise C^0 and it allows “shock” discontinuities. We may, however, relax these conditions by considering piecewise C^1 metric. In this case we may write

$$\partial_\rho G_{\mu\nu} = \partial_\rho G_{\mu\nu}^{(0)} + \gamma_{\mu\nu} u_\rho \theta(u). \quad (17)$$

This expression may be obtained by expanding the metric $G_{\mu\nu}$ in the neighborhood of Σ (as in Eq. (16)) as

$$\begin{aligned} G_{\mu\nu}(x^+) &= G_{\mu\nu 0} + G'_{\mu\nu+}u + \frac{1}{2}G''_{\mu\nu+}u^2 + \dots, \\ G_{\mu\nu}(x^-) &= G_{\mu\nu 0} + G'_{\mu\nu-}u + \frac{1}{2}G''_{\mu\nu-}u^2 + \dots, \end{aligned} \quad (18)$$

The Ricci tensor and the curvature scalar are easily found to be

$$R_{\mu\nu} = R_{\mu\nu}^{(0)} + \frac{1}{2}(u^\kappa u_\nu \gamma_{\mu\kappa} - u_\mu u_\nu \gamma_\kappa^\kappa - u^\kappa u_\kappa \gamma_{\mu\nu} + u_\mu u^\kappa \gamma_{\nu\kappa})\delta(u), \quad (19)$$

$$R = R^{(0)} + (u^\mu u^\nu \gamma_{\mu\nu} - u^\mu u_\mu \gamma_\nu^\nu)\delta(u), \quad (20)$$

with $R_{\mu\nu}^{(0)}, R^{(0)}$ piecewise C^0 . Thus, we may write Eq. (2) as

$$\begin{aligned} &R_{\mu\nu}^{(0)} - \frac{1}{4}H_{\mu\nu\rho}H^{\mu\nu\rho} + 2\nabla_\mu \nabla_\nu \Phi^{(0)} + 2\nabla_\mu(\phi u_\nu)\theta(u) + \\ &\frac{1}{2}(u^\kappa u_\nu \gamma_{\mu\kappa} - u_\mu u_\nu \gamma_\kappa^\kappa - u^\kappa u_\kappa \gamma_{\mu\nu} + u_\mu u^\kappa \gamma_{\nu\kappa})\delta(u) + 2\phi u_\mu u_\nu \delta(u) = 0. \end{aligned} \quad (21)$$

and Eq. (2) has been split into a piecewise C^0 part and a singular part. The only way the above equation to hold is

$$R_{\mu\nu}^{(0)} - \frac{1}{4}H_{\mu\nu\rho}H^{\mu\nu\rho} + 2\nabla_\mu \nabla_\nu \Phi^{(0)} + 2\nabla_\mu(\phi u_\nu)\theta(u) = 0, \quad (22)$$

$$u^\kappa u_\nu \gamma_{\mu\kappa} - u_\mu u_\nu \gamma_\kappa^\kappa - u^\kappa u_\kappa \gamma_{\mu\nu} + u_\mu u^\kappa \gamma_{\nu\kappa} = -4\phi u_\mu u_\nu. \quad (23)$$

Eq. (22) can be solved separately in the two regions $u > 0, u < 0$, while Eq. (23) constrains the possible discontinuities of the metric. In fact, the solution to the latter equation is provided by

$$\gamma_{\mu\nu}u^\nu = \frac{1}{2}\gamma_\nu^\nu u_\mu - 2\phi u_\mu. \quad (24)$$

Thus, shock waves ($\gamma_{\mu\nu} = 0, \phi = 0$) as well as impulsive ones ($\gamma_{\mu\nu} \neq 0, \phi \neq 0$) are both allowed.

Let us now apply the previous results in the case of the collision of plane waves. We will consider only “head on” collision since by making a Lorentz transformation one can arrange the waves to propagate in opposite spatial directions. The metric of a plane wave (in harmonic coordinates) is

$$ds^2 = U(u, x, y)du^2 - 2dudv + dx^2 + dy^2,$$

where

$$U(u, x, y) = f(u)(x^2 - y^2) + g(u)xy.$$

By performing a suitable coordinate transformation, we may write the metric above in the form (Rosen coordinate system)

$$ds^2 = 2e^{-M(u)}dudv + g_{ij}(u)dx^i dx^j, \quad (25)$$

where $i, j = 1, 2$. If the metric g_{ij} of the surface $u = \text{const.}$, $v = \text{const.}$ is diagonalizable by a linear in x^i transformation, the plane wave has constant polarization (i.e., $g(u) = 0$) and this is the type of waves we are dealing with.

Let us now assume that space-time admits a two-parameter abelian group of space-like isometries. The metric for such a space-time can be written as

$$ds^2 = 2e^{-M}dudv + g_{ij}dx^i dx^j, \quad (26)$$

where

$$M = M(u, v) \quad , \quad g_{ij} = g_{ij}(u, v).$$

We may divide space-time into four distinct region labeled as $I : (u < 0, v < 0)$, $II : (u > 0, v < 0)$, $III : (u < 0, v > 0)$, $IV : (u > 0, v > 0)$. Moreover, we will assume that two plane waves in regions II and III moving in opposite spatial directions approach each other in the flat Minkowski region I and, subsequently, they collide in region IV . Thus, the metric in this coordinate system will be of the form (26) with

$$\begin{aligned} I : \quad & M = 0, \quad g_{ij} = \delta_{ij}, \\ II : \quad & M = M(u), \quad g_{ij} = g_{ij}(u), \\ III : \quad & M = M(v), \quad g_{ij} = g_{ij}(v), \\ IV : \quad & M = M(u, v), \quad g_{ij} = g_{ij}(u, v). \end{aligned} \quad (27)$$

The metric in region IV is uniquely determined by a characteristic initial value problem with data determined on the null hypersurfaces $u = 0$ and $v = 0$.

We consider the six-dimensional $SL(2, \mathbb{R})_{k'} \times SU(2)_k$ at levels (k', k) WZW model with central charge

$$c = \frac{3k'}{k' + 2} + \frac{3k}{k + 2}$$

By choosing $k' = -k$, we have $\delta c = \mathcal{O}(1/k^2)$ and the corresponding σ -model will satisfy Eqs. (2–4) with $\delta c = 0$. By gauging an anomaly free two-dimensional abelian subgroup H of $SL(2, \mathbb{R}) \times SU(2)$, one obtains a four-dimensional gauged WZW model with $c = 4 + \mathcal{O}(1/k^2)$

and the target space of the corresponding σ -model could replace Minkowski space-time. One may choose to gauge the $H = \mathbb{R} \times \mathbb{R}$ subgroup which transforms the elements $(g_1, g_2) \in SL(2, \mathbb{R}) \times SU(2)$ as

$$g_1 \rightarrow \exp(\epsilon \sigma_3) g_1 \exp(\bar{\epsilon} \cos \alpha \sigma_3 + \epsilon \sin \alpha \sigma_3), \quad (28)$$

$$g_2 \rightarrow \exp(\bar{\epsilon} \sigma_3) g_2 \exp(i\epsilon \cos \alpha \sigma_2 - i\bar{\epsilon} \sin \alpha \sigma_2), \quad (29)$$

where $(\sigma_i, i = 1, 2, 3)$ are the standard Pauli matrices and α is a free parameter. This coset space has been studied in Ref. [15] and interpreted as a closed inhomogeneous universe [14]. It can also be obtained by an $O(2, 2, \mathbb{R})$ rotation of a product of two dimensional Lorentzian and Euclidean black holes [16], [17]. Parametrizing g_2 as

$$g_2 = \exp(i \frac{\rho + \lambda}{\sqrt{2}} \sigma_2) \exp(i\theta \sigma_3) \exp(i \frac{\rho - \lambda}{\sqrt{2}} \sigma_2)$$

and gauge fixing by choosing

$$g_1 = \begin{pmatrix} \cos \psi & \sin \psi \\ -\sin \psi & \cos \psi \end{pmatrix},$$

one finds that the metric is given by

$$ds^2 = -d\psi^2 + d\theta^2 + G_{\rho\rho} d\rho^2 + G_{\lambda\lambda} d\lambda^2, \quad (30)$$

where

$$G_{\rho\rho} = \frac{4 \cos^2 \psi \cos^2 \theta (1 + \sin \alpha)}{(1 - \cos 2\psi \cos 2\theta) + \sin \alpha (\cos 2\psi - \cos 2\theta)}, \quad (31)$$

$$G_{\lambda\lambda} = \frac{4 \sin^2 \psi \sin^2 \theta (1 - \sin \alpha)}{(1 - \cos 2\psi \cos 2\theta) + \sin \alpha (\cos 2\psi - \cos 2\theta)},$$

The antisymmetric tensor and the dilaton field are found to be

$$B_{\rho\lambda} = \frac{\cos 2\psi - \cos 2\theta + \sin \alpha (1 - \cos 2\psi \cos 2\theta)}{(1 - \cos 2\psi \cos 2\theta) + \sin \alpha (\cos 2\psi - \cos 2\theta)}, \quad (32)$$

$$\Phi = -\frac{1}{2} \ln[1 - \cos 2\psi \cos 2\theta + \sin \alpha (\cos 2\psi - \cos 2\theta)], \quad (33)$$

In terms of advanced and retarded coordinates

$$u = \frac{1}{2}(\psi + \theta) - \frac{\pi}{4} \quad , \quad v = \frac{1}{2}(\psi - \theta),$$

and choosing $\alpha = 0$, the metric, the antisymmetric tensor and the dilaton turn out to be

$$ds^2 = -4dudv + \frac{4\cos^2(u+v+\pi/4)\cos^2(u-v+\pi/4)}{1-\sin[2(u+v)]\sin[2(u-v)]}d\rho^2 + \frac{4\sin^2(u+v+\pi/4)\sin^2(u-v+\pi/4)}{1-\sin[2(u+v)]\sin[2(u-v)]}d\lambda^2, \quad (34)$$

$$B_{\rho\lambda} = \frac{\sin[2(u-v)] - \sin[2(u+v)]}{1 - \sin[2(u+v)]\sin[2(u-v)]}, \quad (35)$$

$$\Phi = -\frac{1}{2}\ln(1 - \sin[2(u+v)]\sin[2(u-v)]). \quad (36)$$

The $\pi/2$ shift in the coordinate u is such that the discontinuities to appear in the surfaces $(u = 0, v = 0)$. On the other hand, the free parameter α has been taken to be zero in order a proper matching with plane waves to be achieved as we will see below. Continuity of the metric, the antisymmetric tensor and the dilaton field across the surface $u = 0$ specifies them in region *III* to be

$$ds_{III}^2 = -4dudv + \frac{\cos^2 2v}{1 + \sin^2 2v}(d\rho^2 + d\lambda^2), \quad (37)$$

$$B_{\rho\lambda}^{III} = -\frac{2\sin 2v}{1 + \sin^2 2v}, \quad (38)$$

$$\Phi^{III} = -\frac{1}{2}\ln(1 + \sin^2 2v), \quad (39)$$

while at the surface $v = 0$ we have

$$ds_{II}^2 = -4dudv + \frac{(\cos u - \sin u)^2}{(\cos u + \sin u)^2}d\rho^2 + \frac{(\cos u + \sin u)^2}{(\cos u - \sin u)^2}d\lambda^2, \quad (40)$$

$$B_{\rho\lambda}^{II} = 0, \quad (41)$$

$$\Phi^{II} = -\ln \cos 2u. \quad (42)$$

Finally, continuity of the metric, antisymmetric tensor and dilaton fields across $v = 0$ of Eqs. (37–39) and/or $u = 0$ of Eqs. (40–42) gives

$$ds_I^2 = -4dudv + d\rho^2 + d\lambda^2, \quad (43)$$

$$B_{\rho\lambda}^I = 0, \quad (44)$$

$$\Phi = 0. \quad (45)$$

Thus, region *I* is flat Minkowski space-time with constant dilaton.

We observe that regions *II*, *III* correspond to plane wave backgrounds moving in opposite directions. After the collision, one expects singularities to be formed as a result of the focusing

effect. In fact, region IV is singular at the surface $u + v + \pi/4 = 0$ which is an orbifold singularity as discussed in Ref. [15].

Let us now examine the type of discontinuities of the derivatives of the metric $G_{\mu\nu}$, the antisymmetric tensor $B_{\mu\nu}$ and the dilaton field Φ .

i) $v=0$:

At the surface $v = 0$ one may check that the derivative of the metric and the dilaton field are continuous, i.e.,

$$\begin{aligned}\partial_u G_{ij}^{IV}|_{v=0} &= \partial_u G_{ij}^{II}|_{v=0}, \\ \partial_v G_{ij}^{IV}|_{v=0} &= \partial_v G_{ij}^{II}|_{v=0}, \\ \partial_u \Phi^{IV}|_{v=0} &= \partial_u \Phi^{II}|_{v=0}, \\ \partial_v \Phi^{IV}|_{v=0} &= \partial_v \Phi^{II}|_{v=0}.\end{aligned}\tag{46}$$

and thus a shock wave discontinuity appears in the $v = 0$ surface. However, the antisymmetric tensor is discontinuous and it is straightforward to check that the corresponding discontinuity in the antisymmetric field strength is of the type (9). Indeed, one may find that

$$\begin{aligned}H_{u\rho\lambda}^{IV} &= \frac{2(\cos[2(u-v)] - \cos[2(u+v)])}{1 - \sin[2(u+v)] \sin[2(u-v)]} + \\ &2 \sin 4u \frac{\sin[2(u-v)] - \sin[2(u+v)]}{(1 - \sin[2(u+v)] \sin[2(u-v)])^2},\end{aligned}\tag{47}$$

$$\begin{aligned}H_{v\rho\lambda}^{IV} &= -\frac{2(\cos[2(u-v)] + \cos[2(u+v)])}{1 - \sin[2(u+v)] \sin[2(u-v)]} + \\ &2 \sin 4v \frac{\sin[2(u-v)] - \sin[2(u+v)]}{(1 - \sin[2(u+v)] \sin[2(u-v)])^2}.\end{aligned}\tag{48}$$

Thus we have

$$\begin{aligned}H_{u\rho\lambda}^{IV}|_{v=0} &= H_{u\rho\lambda}^{II}|_{v=0} = 0, \\ H_{v\rho\lambda}^{IV}|_{v=0} &= \frac{4 \sin 2u}{\cos^2 2u}, \\ H_{v\rho\lambda}^{II}|_{v=0} &= 0\end{aligned}\tag{49}$$

and the discontinuity of the antisymmetric field strength, recalling that $u^\mu = (-1/2, 0, 0, 0)$ for the hypersurface $v = 0$, is of the type (9) with h given by

$$h = -\frac{8 \sin 2u}{\cos^2 2u}.$$

ii) $u=0$:

In the surface $u = 0$ we have

$$\begin{aligned} H_{u\rho\lambda}^{IV}|_{u=0} = H_{u\rho\lambda}^{III} &= 0, \\ H_{v\rho\lambda}^{IV}|_{u=0} = H_{v\rho\lambda}^{III}|_{u=0} &= -\frac{4 \cos 2v}{1 + \sin^2 2v} - \frac{4 \sin 2v \sin 4v}{(1 + \sin^2 2v)^2}, \end{aligned} \quad (50)$$

so that the antisymmetric field strength is continuous across $u = 0$. Furthermore, we find that

$$\begin{aligned} \partial_v G_{ij}^{IV}|_{u=0} &= \partial_v G_{ij}^{III}|_{u=0}, \\ \partial_v \Phi^{IV}|_{u=0} &= \partial_v \Phi^{III}|_{u=0}, \\ \partial_u \Phi^{IV}|_{u=0} &= \partial_u \Phi^{III}|_{u=0}, \end{aligned} \quad (51)$$

However, there exist a discontinuity in the derivative of the metric

$$\begin{aligned} \partial_u G_{\rho\rho}^{IV} &= -\frac{2 \cos 2v}{1 + \sin^2 2v}, \\ \partial_u G_{\lambda\lambda}^{IV} &= \frac{2 \cos 2v}{1 + \sin^2 2v}, \end{aligned} \quad (52)$$

while

$$\partial_u G_{\rho\rho}^{III} = 0 \quad , \quad \partial_u G_{\lambda\lambda}^{III} = 0. \quad (53)$$

Comparing Eqs. (17,52,53) we find that

$$\gamma_{\lambda\lambda} = -\gamma_{\rho\rho} = \frac{2 \cos 2v}{1 + \sin^2 2v} \quad (54)$$

and one may check that the condition (24) is indeed satisfied.

Let us note also that the backgrounds (37–39) and (40–42) have a coset CFT interpretation. To see this, one may check that these backgrounds are singular limits of the $SL(2, \mathbb{R}) \times SU(2)/\mathbb{R} \times \mathbb{R}$ model. Introducing the parameters $\varepsilon, \varepsilon'$ and rescaling the coordinates and the coupling as

$$u \rightarrow \varepsilon u, \quad v \rightarrow \varepsilon' v, \quad \rho \rightarrow \sqrt{\varepsilon \varepsilon'} \rho, \quad \lambda \rightarrow \sqrt{\varepsilon \varepsilon'} \lambda, \quad \alpha' \rightarrow \varepsilon \varepsilon' \alpha', \quad (55)$$

the background (37–39) in region *III* can be obtained in the limit $\varepsilon \rightarrow 0$, the background (40–42) in region *II* in the $\varepsilon' \rightarrow 0$ limit and the flat Minkowski space in *I* when $\varepsilon \rightarrow 0$ and $\varepsilon' \rightarrow 0$.

We have discussed here the conditions which must be satisfied in order the beta-function equations to not be affected by the presence of discontinuities in the target space. We found that

these discontinuities must be across null hypersurfaces and thus gravitational wave backgrounds, shock or impulsive, are exact in string theory. As a result, backgrounds resulting from linear superpositions of independent shock or impulsive waves with parallel propagating wave-fronts are exact as well. We have also discussed the $SL(2, \mathbb{R}) \times SU(2)/\mathbb{R} \times \mathbb{R}$ WZW model and we have shown that this model can be considered as the resulting space-time of the collision of two plane waves. It should be noted, however, that the above leading order solution, indicates what one should expect from a collision of plane waves. Finally, in the present framework, an interesting possibility is the construction of cosmological models built up from gravitational waves [18] where space-time singularities can be understood as a focusing result.

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